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Tables for Facilitating the Determination of Empirical Formulæ.

BY A. W. HALE.

By making the intervals between observations constant and equal to unity,
the calculation of empirical formulæ may be abridged.

Put	$x =$ the independent variable	
	$y =$ " dependent "	
	$x_1, x_2, x_3; y_1, y_2, y_3, \&c.$	= observations
	$m =$ number of observations	
	$a_1, a_2, a_3; b_1, b_2, b_3, \&c.$	= undetermined coefficients
	$[z] = z_1 + z_2 + z_3, \&c.$	
	$[zu] = z_1 u_1 + z_2 u_2 + z_3 u_3, \&c.$	
	$[z^2 u] = z_1^2 u_1 + z_2^2 u_2 + z_3^2 u_3, \&c. \&c.$	
	$x_2 - x_1 = x_3 - x_2 = x_4 - x_3, \&c.$	
	$y = a_1 + a_2 x + a_3 x^2 . . . + a_{n+1} x^n$	
	$u = y - y_1$	(1)
	$z = \frac{x - x_1}{x_2 - x_1}$	(2)
	$u = b_1 z + b_2 z^2 + . . . + b_n z^n$	(3)
	$\left. \begin{aligned} u_1 &= b_1 z_1 + b_2 z_1^2 + . . . + b_n z_1^n \\ u_2 &= b_1 z_2 + b_2 z_2^2 + . . . + b_n z_2^n \\ . & \\ . & \\ . & \\ u_n &= b_1 z_n + b_2 z_n^2 + . . . + b_n z_n^n \end{aligned} \right\}$	(4)

Multiplying equations (4) successively by the coefficients of $b_1, b_2 . . . b_n$ and
adding the results of each multiplication, we have

	$\left. \begin{aligned} [zu] &= [z^2] b_1 + [z^3] b_2 + . . . + [z^{n+1}] b_n \\ [z^2 u] &= [z^3] b_1 + [z^4] b_2 + . . . + [z^{n+2}] b_n \\ . & \\ . & \\ . & \\ [z^n u] &= [z^{n+1}] b_1 + [z^{n+2}] b_2 + . . . + [z^{2n}] b_n \end{aligned} \right\}$	(5)
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Putting $n = 2$ we have from equation (3)

$$u = b_1 z + b_2 z^2 \quad (6)$$

and from equations (5)

$$\begin{cases} [zu] = [z^2] b_1 + [z^3] b_2 \\ [z^2 u] = [z^3] b_1 + [z^4] b_2 \end{cases} \quad (7)$$

Solving equations (7) with respect to b_1 and b_2 and substituting the values of b_1 and b_2 obtained in equation (6) after putting

$$\frac{[z^3]}{[z^2][z^4] - [z^3]^2} = \alpha, \quad \frac{[z^4]}{[z^3]} = \beta, \quad \frac{[z^2]}{[z^3]} = \gamma,$$

we have

$$u = \alpha \{ (\beta [zu] - [z^2 u]) z + (\gamma [z^2 u] - [zu]) z^2 \} \quad (8)$$

By equations (1), (2) and (8) and the following table (1) in which the values of α , β and γ are given for values of m from 5 to 26, a value for y may be found corresponding to any assumed value of x .

Putting $n = 3$ we have from equation (3)

$$u = b_1 z + b_2 z^2 + b_3 z^3 \quad (9)$$

and from equations (5)

$$\begin{cases} [zu] = [z^2] b_1 + [z^3] b_2 + [z^4] b_3 \\ [z^2 u] = [z^3] b_1 + [z^4] b_2 + [z^5] b_3 \\ [z^3 u] = [z^4] b_1 + [z^5] b_2 + [z^6] b_3 \end{cases} \quad (10)$$

Solving equations (10) with respect to b_1 , b_2 and b_3 , after putting

$$\begin{aligned} [z^4]([z^2][z^6] + 2[z^3][z^5] - [z^4]^2) - ([z^2][z^5]^2 + [z^3]^2[z^6]) &= A, \\ [z^3][z^5] - [z^4]^2 &= B, \quad [z^2][z^5] - [z^3][z^4] = C, \quad [z^2][z^4] - [z^3]^2 = D, \\ B/A &= \delta, \quad ([z^4][z^6] - [z^5]^2)/B = \varepsilon, \quad ([z^3][z^6] - [z^4][z^5])/B = \zeta, \quad C/A = \eta, \\ ([z^3][z^6] - [z^4][z^5])/C &= \theta, \quad ([z^2][z^6] - [z^4]^2)/C = \kappa, \quad D/A = \lambda, \quad B/D = \mu, \quad C/D = \nu, \end{aligned}$$

we have

$$\begin{cases} b_1 = \delta(\varepsilon[zu] - \zeta[z^2 u] + [z^3 u]) \\ b_2 = -\eta(\theta[zu] - \kappa[z^2 u] + [z^3 u]) \\ b_3 = \lambda(\mu[zu] - \nu[z^2 u] + [z^3 u]) \end{cases} \quad (11)$$

By equations (1), (2), (9) and (11) and table 2, in which the values of δ , ε , ζ , &c., are given for values of m from 9 to 26, a value for y may be found corresponding to any assumed value of x .

Whether an empirical formula derived from a series of observations will involve the third power of the variable or not, is shown by simply putting the third equation of equations (11)

$$b_3 = \lambda(\mu[zu] - \nu[z^2 u] + [z^3 u]) = 0,$$

whence

$$\nu[z^2u] - \mu[zu] = [z^3u] \quad (12)$$

EXAMPLE 1.

1	2	3	4	5	6	7	8
m	y	x	u	z	zu	z^2u	z^3u
1	5	4	0	0	0	0	0
2	10	6	5	1	5	5	5
3	21	8	16	2	32	64	128
4	38	10	33	3	99	297	891
5	61	12	56	4	224	896	3584
6	90	14	85	5	425	2125	10625
7	125	16	120	6	720	4320	25920
8	166	18	161	7	1127	7889	55223
9	213	20	208	8	1664	13312	106496
					4296	28908	202872
					$[zu]$	$[z^2u]$	$[z^3u]$

In column 1 are given the number of observations m ; in columns 2 and 3 the observations y_1, y_2, x_1, x_2 , &c. The figures in column 4 are calculated from the figures in column 2 by equation (1) by subtracting the value of y for $m = 1$ from the following values of y successively. Column 5 is calculated from column 3 by equation (2) and is in all cases the series of consecutive numbers 0, 1, 2, &c. Columns 6, 7 and 8 are the products of column 5 and columns 4, 6 and 7 respectively.

By equation (12), the footings of columns 6, 7 and 8, example 1, and the values of μ and ν for $m = 9$, Table 2, we have

$$\begin{aligned} \nu \quad [z^2u] \quad \mu \quad [zu] \\ 11\frac{25}{109} \times 28908 - 28\frac{37}{109} \times 4296 = [z^3u] \\ i. e. \quad 202872 \quad = 202872 \end{aligned}$$

The empirical formula for example 1 does not, therefore, involve the third power of the variable.

By equation (8), the footings of columns 6 and 7, example 1, and the values of α , β and γ for $m = 9$, Table 1, we have

$$\begin{aligned} \alpha \quad \beta \quad [zu] \quad [z^2u] \quad \gamma \quad [z^2u] \quad [zu] \\ u = .0118 \{ (6.7685 \times 4296 - 28908)z + (.1574 \times 28908 - 4296)z^2 \} \\ \text{whence} \quad u = 2z + 3z^2 \end{aligned} \quad (13)$$

By equations (1), (2) and (13)

$$y - y_1 = 2 \frac{x - x_1}{x_2 - x_1} + 3 \left(\frac{x - x_1}{x_2 - x_1} \right)^3 \quad (14)$$

By equation (14), example 1, columns 2 and 3 for $m = 1$ and 2, we have

$$y - 5 = 2 \frac{x - 4}{6 - 4} + 3 \left(\frac{x - 4}{6 - 4} \right)^3$$

(15)

whence

$$y = 13 - 5x + \frac{3}{4}x^2$$

EXAMPLE 2.

1	2	3	4	5	6	7	8
m	y	x	u	z	zu	z^2u	z^3u
1	4	3	0	0	0	0	0
2	4	6	0	1	0	0	0
3	4	9	0	2	0	0	0
4	10	12	6	3	18	54	162
5	28	15	24	4	96	384	1536
6	64	18	60	5	300	1500	7500
7	124	21	120	6	720	4320	25920
8	214	24	210	7	1470	10290	72030
9	340	27	336	8	2688	21504	172032
					5292	38052	279180
					$[zu]$	$[z^2u]$	$[z^3u]$

By equation (12), the footings of columns 6, 7 and 8, example 2, and the values of μ and ν for $m = 9$, Table 2, we have

$$\nu \quad [z^2u] \quad \mu \quad [zu]$$

$$11\frac{25}{109} \times 38052, - 28\frac{37}{109} \times 5292, = [z^3u]$$

i. e. $277345\frac{17}{109} < 279180.$

The empirical formula for example 2 therefore involves the third power of the variable.

By equations (11), table 2, for $m = 9$, and the footings of columns 6, 7 and 8, example 2, we have

$$\left. \begin{aligned} b_1 &= .0153(33.5593 \times 5292 - 12.00055 \times 38052 + 279180) = 2, \\ b_2 &= -.0061(30.2857 \times 5292 - 11.5357 \times 38052 + 279180) = -3, \\ b_3 &= .00054(28.33945 \times 5292 - 11.2294 \times 38052 + 279180) = 1 \end{aligned} \right\} \quad (16)$$

Substituting these values of b_1 , b_2 and b_3 in equation (9), we have

$$u = 2z - 3z^2 + z^3 \quad (17)$$

By equations (1), (2) and (17) we have

$$y - y_1 = 2 \frac{x - x_1}{x_2 - x_1} - 3 \left(\frac{x - x_1}{x_2 - x_1} \right)^2 + \left(\frac{x - x_1}{x_2 - x_1} \right)^3 \quad (18)$$

By equation (18) and example 2, columns 2 and 3, for $m = 1$ and 2, we have

$$y - 4 = 2 \frac{x - 3}{6 - 3} - 3 \left(\frac{x - 3}{6 - 3} \right)^2 + \left(\frac{x - 3}{6 - 3} \right)^3$$

whence
$$y = -2 + \frac{11}{3}x - \frac{2}{3}x^2 + \frac{1}{27}x^3 \quad (19)$$

The first and second differential coefficients of y , equation (19), placed equal to zero give a minimum value for y , $y = 3.62$, $x = 7.73$, and a maximum $y = 4.39$, $x = 4.27$, and the point of inflexion $x = 6$, $y = 4$.

In the following tables, repeating decimals are indicated by a dot or bar.

In table 2, in column 2 one, in column 5 two, in column 8 three zeros on the left are omitted, and the tabular numbers in these columns are to be divided by 10, 100, and 1000 respectively.

TABLE I.

1	2	3	4
m	α	β	γ
5	.161290	3.54	.3
6	.069876	4.351	.24
7	.035156	5.15873	.206349
8	.019607	5.96428	.17857142
9	.011795	6.76851	.15740
10	.007520	7.57185	.1407
1	.005020	8.3745	.127
2	.003478	9.176	.116
3	.002486	9.97863	.1068376
4	.001824	10.78022	.098901
5	.001369	11.58158	.0920634
6	.001047	12.3827	.0861
7	.0008149	13.18382	.08088235
8	.0006435	13.984706	.076252723
9	.0005149	14.785575	.072124756
20	.0004169	15.586315	.068421052
1	.0003412	16.386984	.0650793
2	.0002819	17.187590	.062049
3	.0002349	17.988142	.059288537
4	.0002197	18.788647	.056763285
5	.0001670	19.5891	.054
6	.0001423	20.389538	.05230769

TABLE II.

1	2	3	4	5	6	7	8	9	10	11	12
m	10δ	ϵ	ζ	100η	θ	κ	1000λ	μ	μ	ν	ν
9	.152951	33.5593118	12.0005549	.60	30 $\frac{7}{8}$	11.5357142	.5397	28.3 $\frac{7}{8}$	28.3394495	11 $\frac{25}{8}$	11.2293577
10	.086340	42.1331622	13.4445088	.30547	38	12.92	.2429	35 $\frac{1}{8}$	35.5382352	12 $\frac{3}{8}$	12.5735294
1	.051765	51.6587336	14.8854328	.16545	46 $\frac{1}{2}$	14.3015	.1188	43 $\frac{1}{2}$	43.5373493	13 $\frac{1}{8}$	13.9156626
2	.032578	62.1362457	16.3241478	.09496	56	15.681	.06224	52.6 $\frac{7}{8}$	52.3366834	15 $\frac{5}{8}$	15.2562814
3	.021338	73.5658340	17.7612013	.05717	66 $\frac{1}{2}$	17.0604395	.03445	61 $\frac{1}{2}$	61.9861702	16 $\frac{1}{2}$	16.5957446
4	.014452	85.9475854	19.1969727	.03583	77 $\frac{1}{2}$	18.4379905	.01997	72.4 $\frac{1}{2}$	72.3357664	17 $\frac{1}{2}$	17.9343065
5	.010071	99.2815580	20.6317336	.02323	89 $\frac{1}{2}$	19.8146938	.01205	83 $\frac{1}{2}$	83.5354433	19 $\frac{1}{2}$	19.2721518
6	.007193	113.5677917	22.0656837	.01551	102 $\frac{1}{2}$	21.1907142	.007529	95.8 $\frac{5}{8}$	95.5351800	20 $\frac{3}{4}$	20.6094182
7	.005248	128.8063147	23.4989731	.01063	116	22.5661764	.004844	108 $\frac{1}{2}$	108.3349633	21 $\frac{3}{4}$	21.9462102
8	.003902	144.9971474	24.9317168	.00745	130 $\frac{1}{2}$	23.9411764	.003201	121 $\frac{1}{2}$	121.9847826	23 $\frac{1}{2}$	23.2826086
9	.002951	162.1403177	26.3640047	.00532	146	25.3157894	.002164	136.2 $\frac{1}{2}$	136.3346303	24 $\frac{1}{2}$	24.6186770
20	.002264	180.2357981	27.7959078	.00387	162 $\frac{1}{2}$	26.9600751	.001494	151.3 $\frac{3}{8}$	151.5345008	25 $\frac{1}{2}$	25.9544658
1	.001761	199.2836359	29.2274831	.00286	179 $\frac{1}{2}$	28.0640816	.001051	167.1 $\frac{1}{2}$	167.5343898	27 $\frac{1}{2}$	27.2900158
2	.001386	219.2838248	30.6587768	.00215	197 $\frac{1}{2}$	29.4373478	.000752	184.1 $\frac{1}{2}$	184.3342939	28 $\frac{1}{2}$	28.6238602
3	.001103	240.2363700	32.0898267	.00163	216 $\frac{1}{2}$	30.8114059	.0005464	201.1 $\frac{1}{2}$	201.9342105	29 $\frac{1}{2}$	29.9605293
4	.000887	262.1412757	33.5206644	.00126	236	32.1847826	.0004026	220.3 $\frac{1}{2}$	220.3341375	31 $\frac{1}{2}$	31.2955367
5	.000719	284.9985450	34.9513159	.00098	256 $\frac{1}{2}$	33.558	.0003004	239.4 $\frac{1}{2}$	239.5340732	32 $\frac{1}{2}$	32.6304106
6	.000588	308.8081808	36.3818035	.00077	278	34.9310769	.0002268	259.3 $\frac{1}{2}$	259.5340163	33 $\frac{1}{2}$	33.9651657